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A Special One-Sided Approximation Problem

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Let P be a nonnegative perfect spline of degree n on $[a, b]$ satisfying

$$P^{(j)}(a) = P^{(j)}(b) = 0 \qquad (j = 0, ..., n-1).$$

We wish to approximate P from below in the L_1-norm by nonnegative splines of degree $n-1$ with the same fixed knots as P. We show that a unique best approximation exists (differing, in general, from the best approximation without the nonnegativity restriction), and describe the zero structure of the error function. In addition, we discuss the semi-infinite programming approach and deduce relationships with mononsplines and quadrature formulas. © 1990 Academic Press, Inc.

INTRODUCTION

In this paper we consider a special one-sided approximation problem for splines with fixed knots, with resect to the L_1-norm. Such one-sided problems have been considered by several authors, notably Strauß [10], Pinkus [8] and Micchelli and Pinkus [7]. Micchelli and Pinkus prove their results for more general spaces of functions, and consider, as does Strauß, the setting in which the function to be approximated is "generalized convex" with respect to the approximating subspace.

Our problem concerns the approximation from below of a special generalized convex function, a nonnegative perfect spline, and, in contrast to the works cited, we require the approximating splines to be nonnegative as well. The result is that, whereas in the general case the error function exhibits only double zeros, in our case we are forced to deal with higher-order zeros in the knots. Indeed, a major part of Section 1 is devoted to a careful study of these zeros. The appearance of higher-order zeros precludes the possibility of applying the moment theory techniques developed in [7];

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no similar theory involving higher derivatives seems to be available. Thus, our work is closer in nature to that of Strauß.

We are unaware of papers of a similar nature devoted to problems of approximation theory. In the context of semi-infinite programming, the abstract optimization problem has been studied by Krabs [2]. A related problem, that of approximation by linear combinations of functions with nonnegative coefficients, was discussed by Marsaglia [4] and by Krabs [3], and is also treated in the book [2] of Krabs. In our setting, this situation arises only when the approximating splines are piecewise linear ($n=2$). In this case the duality theory for such optimization problems is useful (but not essential) in pinpointing the zeros of the error function. Duality theory may be utilized for $n>2$ as well, to derive equivalent conditions, in terms of positive linear functionals, for the optimal solution. This is done in Section 4.

The connection between one-sided approximation and quadrature formulas has long been known (see, e.g., [7, 10] and the papers cited therein), as has the relation to monosplines. Such a connection exists in our case as well, as is described in Section 5.

The problem considered in this paper arose in the context of shape preserving L_1-approximation [11], the approximation of a continuous function in the L_1-norm from the convex cone of n-convex functions. Using a functional analytic approach, the problem of characterization is reduced to the study of the zeros of a certain perfect spline P. A difficulty arises when P fails to have a full set of zeros. In this case one may approximate P from below by nonnegative splines having the same fixed knots, and substitute for the zeros of P the zeros of the error function. The nonnegativity restriction is essential to these considerations.

In order to solve the one-sided approximation problem it was necessary to develop and/or adapt various tools for dealing with high-order zeros of splines. With few exceptions, this area has been avoided or neglected in the literature of approximation theory—a not uncommon occurrence in pertinent papers is to quote imprecisely the well-known Schoenberg–Whitney Theorem [9]. By ignoring the possibility that, in limiting cases, the relevant determinant can be positive when equality occurs in the interlacing conditions, one risks exchanging rigor for convenience. In extreme cases this may even put the validity of the assertions in doubt. In this paper we have addressed this, and other, fine points of spline theory in the context of the one-sided approximation problem, while striving for generality in the techniques developed to deal with them. Thus, despite the somewhat restricted nature of the problem addressed in these pages, we are hopeful that the results and techniques presented here will find a wider range of applicability than merely to the particular type of problem considered here.

1. Zeros of Splines

In this section we study the zeros of splines of degree $n-1$, and of differences of a perfect spline of degree n and a spline of degree $n-1$. In contrast to the polynomial case, a spline of degree $n-1$ may have zeros of order n without vanishing identically. The presence of knots and the appearance of zero intervals makes a special zero-counting procedure necessary. The main theorem in this section is a kind of Budan–Fourier theorem, giving an upper bound on the number of zeros in an interval when the signs of the derivatives in the endpoints are known. We introduce the concept of a "discretionary zero" and list several corollaries of the main theorem, some, perhaps, of independent interest.

A spline of degree $n-1$ on $[a, b]$ with simple knots $\tau_1 < \cdots < \tau_N$ in (a, b) is a function of the form

$$s(t) = \sum_{i=1}^{N} \alpha_i (\tau_i - t)_+^{n-1} + p_{n-1}(t), \tag{1.1}$$

where p_{n-1} is a polynomial of degree at most $n-1$ and the truncated power $(\tau - t)_+^{n-1}$ is defined as $(\tau - t)^{n-1}$ if $\tau \geqslant t$, and is zero otherwise. For fixed knots $\tau_1, ..., \tau_N$, the $(N+n)$-dimensional linear space of such splines will be denoted by

$$S = S(\tau_1, ..., \tau_N).$$

We will be primarily interested (for reasons that will be made clear in Section 2) in splines that vanish identically outside of (τ_1, τ_N). In this case s is most easily represented as

$$s(t) = \sum_{i=1}^{N-n} c_i M_i(t), \tag{1.2}$$

where M_i is the B-spline based on the knots $\tau_i, ..., \tau_{i+n}$, with support in $[\tau_i, \tau_{i+n}]$ [9]. We define

$$S_0 := \operatorname{span}\{M_1, ..., M_{N-n}\}.$$

The space of polynomials of degree at most $m-1$ (and "ET-spaces" of dimension m in general) have the property that no nontrivial element has more than $m-1$ zeros, counting multiplicities, one less than the dimension of the space. This property is carried over to spaces of splines, provided that the discontinuities in the $(n-1)$st derivative and the presence of zero intervals are taken into account.

Adhering to the zero-counting conventions described in [9], we count the zeros of $s \in S$ and $s \in S_0$ as follows:

(a) Isolated zeros. If $s(\xi) = \cdots = s^{(m-1)}(\xi) = 0$, $s^{(m)}(\xi) \neq 0$, then ξ is a zero of s of order (or multiplicity) m if ξ is an arbitrary point of $[a, b]$ and $m \leqslant n-2$, or if ξ does not coincide with a knot and $m \leqslant n-1$. If $s^{(j)}(\xi) = 0$ ($j = 0, ..., n-2$) then ξ is a zero of multiplicity $n-1$ if $s^{(n-2)}$ changes sign at ξ; if $s^{(n-2)}$ does not change sign at ξ (which can only happen at a knot) then the multiplicity is n.

(b) Endpoint interval zeros. If $s \in S_0$ vanishes on an interval that extends to an endpoint then each subinterval $[\tau_i, \tau_{i+1}] \subseteq [\tau_1, \tau_N]$ on which s vanishes counts as one zero and, for $s \in S$, the interval $[a, \tau_1]$ or $[\tau_N, b]$ counts as n zeros.

(c) Interior interval zeros. Suppose that s vanishes identically on $[\tau_l, \tau_m]$ but is nonzero in at least one point of each of the two neighboring subintervals. Then $[\tau_l, \tau_m]$ counts as $(n+1) + 2\lfloor (m-l-1)/2 \rfloor$ zeros if $s^{(n-2)}$ changes sign in every neighborhood of $[\tau_l, \tau_m]$, and it counts as $n + 2\lfloor (m-l)/2 \rfloor$ zeros otherwise.

Remark. If $s \in S_0$ vanishes identically on $[\tau_j, \tau_{j+1}]$ then, necessarily, s is in

$$\operatorname{span}\{M_1, ..., M_{j-n}, M_{j+1}, ..., M_{N-n}\}.$$

In order for $[\tau_j, \tau_{j+1}]$ to be an interior interval we must have $j - n \geqslant 1$ and $j + 1 \leqslant N - n$, i.e.,

$$n + 1 \leqslant j < j + 1 \leqslant N - n.$$

This shows that the zero-count for interior intervals is consistent with the zero-count for endpoint intervals since there are at least n subintervals $[\tau_i, \tau_{i+1}] \subseteq [\tau_1, \tau_N]$ before $[\tau_j, \tau_{j+1}]$ and at least n such subintervals after $[\tau_j, \tau_{j+1}]$.

(1.3) DEFINITION [9]. $\mathscr{Z}_I(s)$ counts the zeros of a spline s in the interval I according to the procedure outlined above.

(1.4) THEOREM [9]. *Using the counting procedure outlined above,* $\mathscr{Z}_{[\tau_1, \tau_N]}(s) \leqslant N - n - 1$ *for all* $s \in S_0$, $s \not\equiv 0$, *and* $\mathscr{Z}_{[a,b]}(s) \leqslant N + n - 1$ *for all* $s \in S$, $s \not\equiv 0$.

(1.5) DEFINITION. A perfect spline P of degree n on $[a, b]$ with knots $\tau_1 < \cdots < \tau_N$ is a spline of degree n (see (1.1) with $n-1$ replaced by n) such that for $\varepsilon = \pm 1$, $\tau_0 := a$ and $\tau_{N+1} := b$

$$P^{(n)}(t) = \varepsilon(-1)^i, \qquad t \in (\tau_i, \tau_{i+1}) \ (i = 0, ..., N).$$

The set of such perfect splines will be denoted by

$$\mathscr{P} = \mathscr{P}(\tau_1, ..., \tau_N).$$

We now turn to zero-counting for differences of perfect splines of degree n and splines of degree $n-1$.

Let $P \in \mathscr{P}(\tau_1, ..., \tau_N)$ and $s \in S(\tau_1, ..., \tau_N)$ be given. Since $|P^{(n)}| \equiv 1$ in $(a, b) \setminus \{\tau_1, ..., \tau_N\}$ it follows that $P-s$ does not vanish on a subinterval, i.e., it has only isolated zeros. Fix $\xi \in [a, b]$. If

$$(P-s)(\xi) = \cdots = (P-s)^{(m-1)}(\xi) = 0, \qquad (P-s)^{(m)}(\xi) \neq 0,$$

then ξ is a zero of order (or multiplicity) m, provided ξ is arbitrary and $m \leqslant n-2$, or $m \leqslant n$ and ξ is not a knot. If $(P-s)^{(j)}(\xi) = 0$ $(j = 0, ..., n-2)$, and ξ coincides with a knot then ξ is assigned a multiplicity which is one of $n-1, n, n+1$ as follows:

(1.6) If $(P-s)^{(n-2)}$ does not change sign at ξ then the multiplicity is n. If $(P-s)^{(n-2)}$ does change sign at ξ, then the multiplicity is $n-1$ if

$$(P-s)^{(n-1)}_{-}(\xi)(P-s)^{(n-1)}_{+}(\xi) \neq 0;$$

otherwise, it is $n+1$.

Zeros of order greater than $n-2$ will be referred to as *higher-order zeros*.

(1.7) LEMMA. *Let $P \in \mathscr{P}(\tau_1, ..., \tau_N)$ and $s \in S(\tau_1, ..., \tau_N)$ be given. Then for each $0 \leqslant l < m \leqslant N+1$ we have*

$$\mathscr{Z}_{[\tau_l, \tau_m]}(P-s) \leqslant (m-l) + (n-1).$$

Proof. For $n=2$ the lemma is easily verified; in this case zeros of orders 1, 2, and 3 may occur. Applying this result to $(P-s)^{(n-2)}$, we see that it has at most $m-l+1$ zeros on $[\tau_l, \tau_m]$, and the lemma then follows via Rolle's Theorem. ∎

In particular, $P-s$ can have at most $N+n$ sign changes in (a, b). This shows that the space spanned by $S \cup \{P\}$ is a WT-space, i.e., P is *generalized convex* with respect to these splines [7, 12]. This fact, however, will not be explicitly used in the following material.

If a spline (in any of the forms described above) has the maximal number of zeros allowed in a given interval, then we will follow the usual convention and say that it has a *full set* of zeros in that interval.

Suppose that $P-s$ has a higher-order zero in a knot τ_i. Then $(P-s)^{(n-2)}$ vanishes at τ_i and has an inflection point there, going from strict convex to strict concave, or vice versa. This determines the behavior of the one-sided derivatives $(P-s)_{\pm}^{(n-1)}(\tau_i)$. Set

$$p_+ := (P-s)_+^{(n-1)}(\tau_i) \cdot P_+^{(n)}(\tau_i), \qquad p_- := (P-s)_-^{(n-1)}(\tau_i) \cdot P_-^{(n)}(\tau_i).$$

The following situations may occur (see Figures 1(a)–(d):

(1.8) $(P-s)^{(n-2)}$ changes sign at τ_i. Then either

(a) $p_- > 0$ and $p_+ < 0$ (Fig. 1(a)), or

(b) $p_- \leqslant 0$ and $p_+ \geqslant 0$ (Fig. 1(b)).

(1.9) $(P-s)^{(n-2)}$ does not change sign at τ_i. Then either

(a) $P_- > 0$ and $p_+ \geqslant 0$ (Fig. 1(c)), or

(b) $p_- \leqslant 0$ and $p_+ < 0$ (Fig. 1(d)).

The following definition is useful in describing these situations.

(1.10) DEFINITION. Let $P-s$ have a higher-order zero in a knot τ_i. If $p_- \leqslant 0$ then $(P-s)^{(n-1)}$ has a *discretionary zero* in τ_i; if $p_+ \geqslant 0$ then

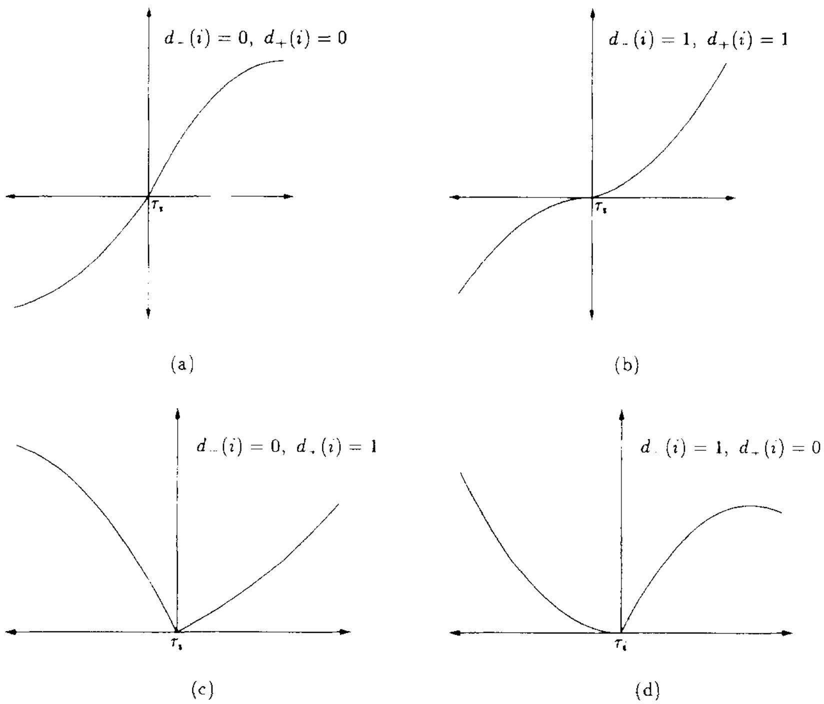

FIGURE 1

$(P-s)^{(n-1)}_-$ has a discretionary zero there. If $(P-s)^{(n-1)}_-$ or $(P-s)^{(n-1)}_+$ vanishes at a discretionary zero we say that the zero is *taken on*. The number of discretionary zeros (either 0 or 1) of $(P-s)^{(n-1)}_+$ and $(P-s)^{(n-1)}_-$ in τ_i is denoted by $d_+(i)$ and $d_-(i)$, respectively.

Thus, $d_+(i)=1$ iff $p_+ \geqslant 0$ and $d_-(i)=1$ iff $p_- \leqslant 0$.

Figures 1(a)–(d) show possible configurations of $(P-s)^{(n-2)}$ if $P-s$ has a higher-order zero in τ_i, $1 \leqslant i \leqslant N$, with the corresponding values of $d_+(i)$ and $d_-(i)$.

A much more precise statement may be made about the zeros of $P-s$ on open intervals, given information about the zeros in the endpoints. In order to prove the main theorem of this section, a theorem of Budan–Fourier type, we first recall the following notation.

(1.11) DEFINITION. $S^+(x_0, ..., x_n)$ denotes the maximal number of sign changes in the sequence $x_0, ..., x_n$, where a zero entry is allowed to be $+1$ or -1.

The next lemma follows immediately from Definitions (1.10) and (1.11) (see also Figures 1(a)–(d)).

(1.12) LEMMA. *Let P and s be as in* (1.7), *and assume that $P-s$ has a higher-order zero in τ_i, $0 \leqslant i \leqslant N+1$. Then*

$$d_+(i) = S^+((P-s)^{(n-1)}_+(\tau_i), P^{(n)}_+(\tau_i)), \textit{ and}$$

$$d_-(i) = S^+((-1)^{n-1}(P-s)^{(n-1)}_-(\tau_i), (-1)^n P^{(n)}_-(\tau_i)).$$

We now record several properties of $S^+(x_0, ..., x_n)$ that will be used in the proof of (1.13).

(i) If $x_0 = \cdots = x_j = 0$ then

$$S^+(x_0, ..., x_n) = j+1+S^+(x_{j+1}, ..., x_n).$$

(ii) If $x_j \neq 0$ then

$$S^+(x_0, ..., x_n) = S^+(x_0, ..., x_j) + S^+(x_j, ..., x_n).$$

(iii) If $x_j = 0$ then

$$S^+(x_0, ..., x_n) \geqslant S^+(x_0, ..., x_j) + S^+(x_j, ..., x_n) - 1.$$

(iv) For all $x_0, ..., x_n$

$$S^+(x_0, ..., x_n) + S^+(x_0, -x_1, ..., (-1)^n x_n) \geqslant n.$$

If $x_n = 0$ then

$$S^+(x_0, \ldots, x_n) + S^+(x_0, -x_1, \ldots, (-1)^n x_n) \geqslant n+1.$$

The following theorem provides the kind of precise zero-counting needed to prove the main theorem of this paper.

(1.13) THEOREM. *For $P \in \mathscr{P}(\tau_1, \ldots, \tau_N)$ and $s \in S(\tau_1, \ldots, \tau_N)$ we have*

$$\begin{aligned} \mathscr{Z}_{(a,b)}(P-s) \leqslant N + n - S^+((P-s)\ (b), \ldots, (P-s)^{(n)}(b)) \\ - S^+((P-s)_+(a), -(P-s)'_+(a), \ldots, (-1)^n (P-s)^{(n)}_+(a)). \end{aligned} \tag{1.14}$$

Proof. We first introduce the following convenient notation: For fixed τ_i and $0 \leqslant l < m \leqslant n$, set

$$\begin{aligned} S_i^+(l, m) := S^+((P-s)^{(l)}(\tau_i), \ldots, (P-s)^{(m)}(\tau_i)) \\ + S^+((-1)^l (P-s)^{(l)}_+(\tau_i), \ldots, (-1)^m (P-s)^{(m)}_+(\tau_i)). \end{aligned}$$

The proof initially follows the lines of [6, Proposition 1]: We apply the Budan–Fourier theorem for polynomials to $(P-s)|_{(\tau_i, \tau_{i+1})}$ for each i, then sum over i and include the zeros at τ_i of multiplicity, say, m_i. We observe that the left-hand side of (1.14) differs from the right-hand side by a quantity $\sum_{i=1}^N T_n(\tau_i)$, where $T_n(\tau_i) := S_i^+(0, n) - m_i - n + 1$, and the proof is completed by showing

$$S_i^+(0, n) \geqslant m_i + n - 1. \tag{1.15}$$

Suppose first that m_i satisfies $m_i \leqslant n-2$. Then, by property (i) above, $S_i^+(0, n) = 2m_i + S_i^+(m_i, n)$. If $(P-s)^{(n-2)}(\tau_i)$ is nonzero, property (ii) yields

$$S_i^+(m_i, n) = S_i^+(m_i, n-2) + S_i^+(n-2, n); \tag{1.16}$$

otherwise, (iii) yields

$$S_i^+(m_i, n) \geqslant S_i^+(m_i, n-2) + S_i^+(n-2, n) - 2. \tag{1.17}$$

Thus, if $(P-s)^{(n-2)}(\tau_i)$ is nonzero, (1.16) and property (iv) give

$$\begin{aligned} S_i^+(0, n) &= 2m_i + S_i^+(m_i, n-2) + S_i^+(n-2, n) \\ &\geqslant m_i + n - 2 + S_i^+(n-2, n). \end{aligned}$$

One easily checks that $P^{(n)}(\tau_i)\, P^{(n)}_+(\tau_i) < 0$ implies $S_i^+(n-2, n) \geqslant 1$, hence

(1.15) is valid. If $(P-s)^{(n-2)}(\tau_i)=0$, then from property (i) we have $S_i^+(n-2, n)=2+S_i^+(n-1, n)$, hence from (1.17) we get

$$S_i^+(0, n) \geqslant 2m_i + S_i^+(m_i, n-2) + S_i^+(n-1, n)$$
$$\geqslant m_i + n - 1 + S_i^+(n-1, n) \geqslant m_i + n - 1.$$

Therefore, (1.15) is valid here, too.

Now suppose that τ_i is a higher-order zero. Then $(P-s)^{(j)}(\tau_i)=0$ $(j=0, \dots, n-2)$, hence from property (i) we have

$$S_i^+(0, n) = 2(n-1) + S_i^+(n-1, n). \tag{1.18}$$

It thus suffices to show

$$m_i \leqslant n - 1 + S_i^+(n-1, n). \tag{1.19}$$

Note that (1.12) implies

$$S_i^+(n-1, n) = d_+(i) + d_-(i).$$

By our definition of higher-order zeros we always have

$$m_i \leqslant n - 1 + d_-(i) + d_+(i),$$

thus (1.19) holds, and the proof is complete.

We now list several corolllaries of (1.13). Applying the proof of (1.13) on the interval $[\tau_l, \tau_m]$ yields

(1.20) COROLLARY. *Let P and s be as in* (1.13). *Then for $0 \leqslant l < m \leqslant N+1$ we have*

$$\mathscr{Z}_{(\tau_l, \tau_m)}(P-s) \leqslant m - l + n - 1 - S^+((P-s)_-(\tau_m), \dots, (P-s)^{(n)}(\tau_m))$$
$$- S^+((P-s)_+(\tau_l), -(P-s)'_+(\tau_l), \dots, (-1)^n (P-s)^{(n)}_+(\tau_l)).$$

An important consequence of (1.20) is that the maximal number of zeros of $P-s$ on an open interval bounded by higher-order zeros is independent of whether or not any discretionary zeros are taken on.

(1.21) COROLLARY. *Let P and s be as in* (1.13), *and assume that $P-s$ has a higher-order zero in τ_l and in τ_m $(0 \leqslant l < m \leqslant N+1)$ Then*

$$\mathscr{Z}_{(\tau_l, \tau_m)}(P-s) \leqslant m - l - n + 1 - d_-(m) - d_+(l)$$

Consequently,

$$m \geqslant l + n - 1 + d_-(m) + d_+(l).$$

(1.22) COROLLARY. *Let P and s be as in* (1.13). *If $P-s$ satisfies*

$$(P-s)_+^{(j)}(a)=(P-s)^{(j)}(b)=0 \qquad (j=0, ..., n-1) \tag{1.23}$$

and $P-s$ has a higher-order zero in τ_i then

$$n+d_-(i) \leqslant i \leqslant N-n+1-d_+(i).$$

Corollaries (1.21) and (1.22) show that if $P-s$ has several higher-order zeros then these zeros must be sufficiently separated. This should be kept in mind when such assumptions are made in Section 2.

2. THE ONE-SIDED APPROXIMATION PROBLEM

In this section P_0 will be a fixed element of $\mathscr{P}(\tau_1, ..., \tau_N)$ satisfying

$$P_0 \geqslant 0 \text{ in } [a, b], \qquad \text{and} \qquad P_0^{(j)}(a)=P_0^{(j)}(b)=0 \; (j=0, ..., n-1). \tag{2.1}$$

Conditions (2.1) imply that $P_0(t)=(1/n!)(t-a)^n$ in $[a, \tau_1]$ and $P_0(t)=(1/n!)(b-t)^n$ in $[\tau_N, b]$, from which it follows that P_0 does not vanish in $(a, \tau_1] \cup [\tau_N, b)$. Moreover, since $P_0^{(n)} \equiv 1$ in (a, τ_1) and $(-1)^n P_0^{(n)} \equiv 1$ in (τ_N, b), we conclude that $\varepsilon = 1$ in Definition (1.5) and that $N-n$ is even.

We wish to approximate P_0 from below in the L_1-norm by nonnegative elements of S. Since $0 \leqslant s \leqslant P_0$ implies $s \equiv 0$ outside of $[\tau_1, \tau_N]$ (due to conditions (2.1)), we may restrict our attention to $s \in S_0$. Thus, our problem may be expressed as

$$\text{Minimize} \int_a^b (P_0 - s) \text{ subject to } s \in S_P := \{s \in S_0 : 0 \leqslant s \leqslant P_0\}.$$

In this section we prove the existence and uniqueness of the best L_1-approximation $s_0 \in S_P$ to P_0. While the existence follows from elementary compactness considerations, the proof of uniqueness is complicated by the possibility of higher-order zeros of the error function $P_0 - s_0$ in the knots. The standard procedure, which we follow in principle, is to show that the error function must have a full set of $N+n$ zeros in $[a, b]$ if $s_0 \in S_P$ is a best approximation. One then shows that if s_1 is another best approximation then not only does $P_0 - s_1$ have $N+n$ zeros, but $P_0 - s_0$ and $P_0 - s_1$ share a set of $N+n$ zeros, counting multiplicities. Therefore, $s_1 - s_0$ has $N+n$ zeros in $[a, b]$ (and $N-n$ in (τ_1, τ_N)), which is only possible if $s_1 \equiv s_0$. The problem with this argument is that if a higher-order zero coincides with a knot, the difference $s_1 - s_0$ need not inherit this zero with as great a multiplicity. To see an example of this phenomenon, we need only consider two splines, s_0 and s_1, of degree 1 that share a double zero in a knot τ_i (i.e., they vanish at τ_i and do not change sign there). If $s := s_1 - s_0$ changes sign at τ_i then s has only a simple zero in τ_i. This shows that, in contrast to polynomials, two splines may share a full set of zeros without their difference vanishing identically.

Hence, in order to prove uniqueness we must not only show that the error function for the best approximation has a full set of zeros, we must also consider what happens if higher-order zeros occur in knots. The bulk of the work in this section is therefore devoted to this task.

The next two lemmas are simple consequences of Taylor's Theorem. For completeness we prove the second of the two.

(2.2) LEMMA. *Let $f, g \in C^j[a, b]$ be given and let f have a zero $\xi \in (a, b)$ of even order $m \leqslant j$. If $0 \leqslant g \leqslant f$ holds in $[a, b]$ then g vanishes in ξ with multiplicity at least m. If j is even and $f^{(i)}(\xi) = 0$ $(i = 0, ..., j)$, then this holds for g as well.*

(2.3) LEMMA. *Suppose that $f \in C^j[a, b]$ vanishes in $\xi_1, ..., \xi_s \in (c, d)$ with exact even orders $m_i \leqslant j$, and that $g \in C^j[a, b]$ has zeros in ξ_i with at least the same multiplicities. If, additionally, f and g are nonnegative in $[a, b]$ and f has no other zeros there, then for some $c > 0$ we have*

$$cg \leqslant f \qquad \text{in } [a, b].$$

Proof. For simplicity, we suppose that $s = 1$. If (2.4) does not hold then for each $k = 1, 2, ...$ there is a point $t_k \in [a, b]$ such that

$$\frac{1}{k} g(t_k) > f(t_k) \geqslant 0. \tag{2.5}$$

By going to a subsequence we may assume that $t_k \to t_0 \in [a, b]$, hence from (2.5) we get $0 \geqslant f(t_0)$, i.e., $f(t_0) = 0$. Thus, $t_0 = \xi_1$. Expanding (2.5) about ξ_1 and using the fact that ξ_1 is a zero of even multiplicity m_1 yields

$$\frac{1}{k} g^{(m_1)}(\xi_1 + \theta_k (t_k - \xi_1)) \geqslant f^{(m_1)}(\xi_1 + \hat{\theta}_k (t_k - \xi_1)) \geqslant 0,$$

with $0 < \theta_k, \hat{\theta}_k < 1$. Thus, as $k \to \infty$ we get $f^{(m_1)}(\xi_1) = 0$, a contradiction to our assumption that ξ_1 is a zero of f of exact order m_1.

(2.6) DEFINITION. For fixed $1 \leqslant l < m \leqslant N$

$$S_{0(l,m)} \subseteq S_0$$

denotes the subspace of splines vanishing outside of $[\tau_l, \tau_m]$. If $m-n \geqslant l$ then $S_{0(l,m)}$ is spanned by $\{M_l, ..., M_{m-n}\}$ and thus has dimension $m-l-n+1$.

The following lemma follows directly from (1.4).

(2.7) LEMMA. *Suppose that, for $s \in S_0$, $P_0 - s$ has higher-order zeros in τ_l and in τ_m $(0 \leqslant l < m \leqslant N+1)$ and has a full set of zeros in (τ_l, τ_m). Set $\hat{l} := l + d_+(l)$, $\hat{m} := m - d_-(m)$. If $u \in S_{0(\hat{l},\hat{m})}$ vanishes at the zeros of $P_0 - s$ in (τ_l, τ_m), with the same multiplicities, then u vanishes identically.*

Proof. We note that if $P_0 - s$ has a full set of zeros in (τ_l, τ_m) then

$$\mathscr{Z}_{(\tau_l,\tau_m)}(P_0 - s) = m - l - n + 1 - d_+(l) - d_-(m) = \dim S_{0(\hat{l},\hat{m})}.$$

It thus follows from (1.4) that any element of $S_{0(\hat{l},\hat{m})}$ that vanishes at the zeros of $P_0 - s$, with the same multiplicities, vanishes identically.

Our next lemma shows how to construct nonnegative elements of $S_{0(l,m)}$ with a given number of zeros.

(2.8) LEMMA. *Let points $\tau_l < \xi_1 < \cdots < \xi_s < \tau_m$ and even integers m_i be given, such that $2 \leqslant m_i \leqslant n-2$ if $\xi_i \in \{\tau_{l+1}, ..., \tau_{m-1}\}$ and otherwise $2 \leqslant m_i \leqslant n$. Assume that $\sum_{i=1}^{s} m_i \leqslant m - l + n$. Then there is a nonnegative, nontrivial element $u \in S_{0(l,m)}$ vanishing at ξ_i with multiplicity (at least) m_i $(i = 1, ..., s)$.*

Proof. To construct u we "smooth" $M_l, ..., M_{m-n}$ by convolution with the Gaussian kernel (see, e.g., [7]) with parameter $\varepsilon > 0$. In the transformed space span$\{M_l^\varepsilon, ..., M_{m-n}^\varepsilon\}$ of analytic functions we may uniquely define elements with $m-n-l+1$ function and successive derivative values (the smoothed space is an "ET-space"). In particular, we may define a nonnegative element $u_\varepsilon = \sum_{j=l}^{m-n} a_j^\varepsilon M_j^\varepsilon$ with zeros at ξ_i of multiplicity m_i. By requiring $\sum |a_j^\varepsilon| = 1$ for all $\varepsilon > 0$ we guarantee that a limit point exists as $\varepsilon \downarrow 0$ (through a subsequence) and that the resulting function u is nontrivial. Moreover, since $\{u_\varepsilon^{(j)}\}$ converges uniformly on $[\tau_l, \tau_m]$ to $u^{(j)}$ for $0 \leqslant j \leqslant n-2$, and $\{u_\varepsilon^{(n-1)}\}$ converges uniformly on compact subsets of $(\tau_l, \tau_m) \backslash \{\tau_{l+1}, ..., \tau_{m-1}\}$ to $u^{(n-1)}$, it follows that u has the zeros ξ_i with multiplicities at least m_i. ∎

(2.9) LEMMA. *Suppose that, for $s \in S_P$, $P_0 - s$ has higher-order zeros in τ_l and in τ_m $(0 \leqslant l < m \leqslant N+1)$, and has no zeros of higher order in any of the knots $\tau_{l+1}, ..., \tau_{m-1}$. If $P_0 - s$ fails to have a full set of zeros in $[\tau_l, \tau_m]$ then there is a nontrivial $u \in S_{0(l,m)}$ with $0 \leqslant u \leqslant P_0 - s$ in $[a, b]$.*

Proof. Let $\hat{l} := l + d_+(l)$ and $\hat{m} := m - d_-(m)$. From (1.21) we have

$$\mathscr{Z}_{(\tau_l, \tau_m)}(P_0 - s) \leqslant m - l - n + 1 - d_+(l) - d_-(m) = \dim S_{0(\hat{l}, \hat{m})}.$$

Suppose first that $d_+(l) = d_-(m) = 0$. If $(P_0 - s)|_{[\tau_l, \tau_m]}$ fails to have a full set of zeros then it must lose at least two zeros in (τ_l, τ_m), i.e.,

$$\mathscr{Z}_{(\tau_l, \tau_m)}(P_0 - s) \leqslant m - l - n - 1.$$

Lemma (2.8) guarantees the existence of a nonnegative, nontrivial $v \in S_{0(l,m)}$ with at least the zeros of $P_0 - s$ in (τ_l, τ_m), counting multiplicities, and, from (2.3), there is a $c > 0$ for which $u := cv \leqslant P_0 - s$ holds in $[\tau_l, \tau_m]$. Since $u \equiv 0$ outside of $[\tau_l, \tau_m]$, we have $0 \leqslant u \leqslant P_0 - s$ on all of $[a, b]$.

The other cases will be treated in a similar way. For the sake of brevity we only consider the case in which $d_-(m) = d_+(l) = 1$, the most difficult of the remaining cases. If $(P_0 - s)|_{[\tau_l, \tau_m]}$ fails to have a full set of zeros then either both discretionary zeros are taken on and $P_0 - s$ loses at least two zeros in (τ_l, τ_m), or at least one of the discretionary zeros is not taken on. In the first case we have

$$\mathscr{Z}_{(\tau_l, \tau_m)}(P_0 - s) \leqslant m - l - n - 3 \tag{2.10}$$

and in the second case we have

$$\mathscr{Z}_{(\tau_l, \tau_m)}(P_0 - s) \leqslant m - l - n - 1. \tag{2.11}$$

When (2.10) is valid we construct u as before, but in the $(m - l - n - 1)$-dimensional space $S_{0(\hat{l}, \hat{m})}$, in which nontrivial elements may have up to $m - l - n - 2$ zeros; when (2.11) holds we construct u in either $S_{0(l+1, m)}$ or in $S_{0(l, m-1)}$, depending on whether the discretionary zero in τ_l or τ_m is taken on. In either case the dimension is $m - l - n$ and we may require u to have at most $m - l - n - 1$ zeros. The proof is now advanced as before, with the aid of (2.3). ∎

We are now prepared to prove the main theorem of this paper.

(2.12) THEOREM. *P_0 has a unique best L_1-approximation s_0 from S_P. The error function $P_0 - s_0$ has a full set of zeros in $[a, b]$, and if any higher-order zeros of $P_0 - s_0$ coincide with knots, then all discretionary zeros of $(P_0 - s)^{(n-1)}_-$ and $(P_0 - s)^{(n-1)}_+$ are taken on.*

Proof. Because S_0 is finite-dimensional and S_P is nonempty ($s \equiv 0$ is in S_P), an elementary compactness argument yields an element $s_0 \in S_P$ that minimizes $\int_a^b (P_0 - s)$.

From (2.9) it follows that $P_0 - s_0$ must have a full set of zeros on each interval $[\tau_l, \tau_m]$ $(0 \leqslant l < m \leqslant N + 1)$ bounded by zeros of higher order, in which no other higher-order zeros coincide with knots. For, if this were not the case, we could find a nontrivial $u \in S_0$ satisfying $0 \leqslant u \leqslant P_0 - s_0$. But then $s_0 + u$ would be a better approximation to P_0 from S_P, in contradiction to the optimality of s_0. In particular, $P_0 - s_0$ must have a full set of zeros in $[a, b]$ and all discretionary zeros must be taken on.

Now suppose that $s_1 \in S_P$ is also a best approximation to P_0. Then $s := \frac{1}{2}(s_0 + s_1)$ is also best since

$$P_0 - s = \tfrac{1}{2}(P_0 - s_0) + \tfrac{1}{2}(P_0 - s_1). \tag{2.13}$$

The nonnegativity of the terms in (2.13) implies

$$P_0 - s \geqslant \tfrac{1}{2}(P_0 - s_i) \qquad (i = 0, 1),$$

hence from (2.2), and from the definition of the higher-order zeros, both of $P_0 - s_0$ and $P_0 - s_1$ share the zeros of $P_0 - s$ (and have no other zeros, since all three have a full set). If $P_0 - s$ has no higher-order zeros that coincide with knots then we easily see that $s_1 - s_0$ has at least $N - n$ zeros in (τ_1, τ_N), and therefore $s_1 \equiv s_0$. Otherwise, we restrict our attention to intervals $[\tau_l, \tau_m]$ $(0 \leqslant l < m \leqslant N + 1)$ bounded by higher-order zeros of $P_0 - s$, in which no other higher-order zeros coincide with knots. On each of these intervals $P_0 - s$ has a full set of zeros, which are inherited by $s_1 - s_0$ with their full multiplicities. Hence, from (1.4), $s_1 - s_0$ must vanish identically. Thus, $s_1 \equiv s_0$ on all of $[a, b]$ and uniqueness has been shown. This completes the proof of the theorem.

Remark. It is an interesting consequence of (2.12) that if $P_0 - s_0$ has a higher-order zero in a knot τ_i for which $d_-(i) = d_+(i) = 1$, then $(P_0 - s)^{(n-1)}(\tau_i) = 0$, so that $s_0^{(n-1)}$ is continuous in τ_i. Thus, if i is even and $P_0 - s_0$ has a higher-order zero in τ_i, then the coefficient of $(\tau_i - t)_+^{(n-1)}$ in the expansion (1.1) must vanish (this behavior is already present in the case $n = 1$, where a piecewise linear perfect spline is approximated by piecewise constant splines).

3. Determinants and Interlacing Conditions

In this section we show that the bounds on the number of zeros derived in Section 1 imply that the zeros and the knots of $P_0 - s_0$ must "interlace" in a certain manner. We then apply the Schoenberg–Whitney Theorem to prove that a matrix involving these knots and zeros is nonsingular. Certain modifications must be made to the determinant that one usually encounters in such situations, in case higher-order zeros coincide with knots.

(3.1) Lemma. *Let $P \in \mathscr{P}$ and $s \in S$ be given, and let $P-s$ satisfy* (1.23). *If $P-s$ has zeros $a < \xi_1 < \cdots < \xi_r < b$ with multiplicities $m_i \leq n+1$, such that $n + \sum_{i=1}^{r} m_i = N$, then*

$$\tau_{l(i)} \leq \xi_i \leq \tau_{l(i-1)+n+1} \qquad (i = 1, ..., r), \tag{3.2}$$

where $l(i) := \sum_{j=1}^{i} m_j$, $l(0) := 0$. Equality holds in (3.2) *only if*

(a) $m_i = n+1$ *and* $\xi_i = \tau_{l(i)} = \tau_{l(i-1)+n+1}$,

(b) $m_i = n$, $\xi_i = \tau_{l(i)}$, *and* $d_-(l(i)) = 0$, *or*

(c) $m_i = n$, $\xi_i = \tau_{l(i-1)+n+1} = \tau_{l(i)+1}$, *and* $d_+(l(i)+1) = 0$.

Proof. All of these assertions are consequences of (1.7) and (1.21). Suppose, for example, that $\xi_i < \tau_{l(i)}$. Then $P-s$ has at least $n + \sum_{j=1}^{i} m_j = n + l(i)$ zeros in $[a, \tau_{l(i)}]$. However, by (1.7),

$$\mathscr{Z}_{[\tau_0, \tau_{l(i)}]}(P-s) \leq l(i) + n - 1,$$

hence there is a contradiction. The same contradiction results if $\xi_i = \tau_{l(i)}$ and $m_i \leq n-1$. If $\xi_i = \tau_{l(i)}$, $m_i = n$ and $d_-(l(i)) = 1$, then (1.21) asserts that

$$\mathscr{Z}_{(\tau_0, \tau_{l(i)})}(P-s) \leq l(i) - n + 1 - d_+(0) - d_-(l(i)) = l(i) - n - 1 = l(i-1) - 1.$$

But $P-s$ has at least $\sum_{j=1}^{i-1} m_j = l(i-1)$ zeros in $(a, \tau_{l(i)})$, again a contradiction. An analogous argument shows that $\xi_i < \tau_{l(i-1)+n+1}$ if $m_i \leq n-1$, and if $m_i = n$ then $\xi_i = \tau_{l(i-1)+n+1} = \tau_{l(i)+1}$ is allowed only if $d_+(l(i)+1) = 0$.

Now suppose that $m_i = n+1$. As before, $\xi_i < \tau_{l(i)} = \tau_{l(i-1)+n+1}$ implies that $P-s$ has at least $n + l(i)$ zeros in $[a, \tau_{l(i)}]$, a contradiction to (1.7), and $\xi_i > \tau_{l(i)}$ implies that $P-s$ has at least $n + \sum_{j=i}^{r} m_j = N - l(i) + n + 1$ zeros in $[\tau_{l(i)}, b]$, in contradiction to (1.7). Thus, $\xi_i = \tau_{l(i)}$ (no contradiction arises here since ξ_i counts as at most n zeros on each of $[a, \tau_{l(i)}]$ and $[\tau_{l(i)}, b]$). ▮

(3.3) Theorem. *Let s_0 be the best approximation to P_0 found in* (2.12), *and let $P_0 - s_0$ have the zeros $a < \xi_1 < \cdots < \xi_r < b$ with multiplicities m_i $(i = 1, ..., r)$. Then the "interlacing conditions"* (3.2) *hold, with equality only if*

(a) *n is odd and $m_i = n+1$, in which case $\xi_i = \tau_{l(i)} = \tau_{l(i-1)+n+1}$, or*

(b) *n is even and $m_i = n$, in which case both $\xi_i = \tau_{l(i)}$ and $\xi_i = \tau_{l(i-1)+n+1} = \tau_{l(i)+1}$ are allowed.*

Proof. Since s_0 is in S_0 and P_0 satisfies (2.1), the assumptions of (3.1) are satisfied and therefore the inequalities (2.12) are valid. Since all m_i are even, $m_i = n+1$ only if n is odd, thus (a) follows from (a) of (3.1).

Similarly, if $m_i = n$ then n must be even. In this case, if $\xi_i = \tau_{l(i)}$ then, since $l(i)$ is even, $d_+(l(i)) = 1$ and $d_-(l(i)) = 0$, and if $\xi_i = \tau_{l(i)+1}$ then, since $l(i)+1$ is odd, $d_-(l(i)+1) = 1$ and $d_+(l(i)+1) = 0$. These are in accordance with (b) and (c) of (3.1).

(3.4) DEFINITION. Let $a < \xi_1 < \cdots < \xi_r$ and $m_i \leqslant n+1$ be given, with $n + \sum_{i=1}^r m_i = N$, and suppose that the interlacing conditions (3.2) are met. The determinant

$$\begin{pmatrix} 0, \ldots, n-1, \xi_1, \ldots, \xi_r \\ \tau_1, \ldots, \tau_N \end{pmatrix}$$

is defined as follows:

(i) If $m_i \leqslant n-1$ for all $i = 1, \ldots, r$ then the determinant is defined as

$$\begin{vmatrix} 1 & \cdots & 1 \\ \tau_1 & \cdots & \tau_N \\ \vdots & & \vdots \\ \tau_1^{n-1} & \cdots & \tau_N^{n-1} \\ (\tau_1-\xi_1)_+^{n-m_1} & \cdots & (\tau_N-\xi_1)_+^{n-m_1} \\ \vdots & & \vdots \\ (\tau_1-\xi_1)_+^{n-1} & \cdots & (\tau_N-\xi_1)_+^{n-1} \\ \vdots & & \vdots \\ (\tau_1-\xi_r)_+^{n-m_r} & \cdots & (\tau_N-\xi_r)_+^{n-m_r} \\ \vdots & & \vdots \\ (\tau_1-\xi_r)_+^{n-1} & \cdots & (\tau_N-\xi_r)_+^{n-1} \end{vmatrix}. \tag{3.5}$$

(ii) If $m_i = n$ for some $1 \leqslant i \leqslant r$ (so that $\tau_{l(i)} \leqslant \xi_i \leqslant \tau_{l(i)+1}$) then the determinant is defined as above, with the conventions

$$\xi_i = \tau_{l(i)} \Rightarrow (\tau_{l(i)} - t)_+^0 \,|_{t=\xi_i} := 0,$$

$$\xi_i = \tau_{l(i)+1} \Rightarrow (\tau_{l(i)+1} - t)_+^0 \,|_{t=\xi_i} := 1.$$

(iii) If $m_i = n+1$ for some $1 \leqslant i \leqslant r$ (so that $\tau_{l(i)} = \xi_i = \tau_{l(i-1)+n+1}$) then the determinant is defined as above, but with m_i set to n and the column with $\tau_{l(i)}$ deleted.

(3.6) THEOREM. *Let $P \in \mathscr{P}$ and $s \in S$ satisfy (1.23) and let $a < \xi_1 < \cdots < \xi_r < b$ be the zeros of $P - s$ with multiplicities $m_i \leqslant n+1$ such that $n + \sum_{i=1}^r m_i = N$. Then the determinant*

$$D := \begin{pmatrix} 0, \ldots, n-1, \xi_1, \ldots, \xi_r \\ \tau_1, \ldots, \tau_N \end{pmatrix}$$

defined in (3.4) is positive.

Proof. If all m_i are at most $n-1$ then this is a standard consequence of the Schoenberg–Whitney Theorem (see, e.g., [9]), which states that D is nonzero (i.e., positive) iff

$$x_i \in (\tau_i, \tau_{i+n}) \qquad (i = 1, ..., N-n),$$

where $x_1 \leqslant \cdots \leqslant x_{N-n}$ are the ξ_i repeated according to their multiplicities. We now show that if $m_i = n$ or $m_i = n+1$ for some i then D may be factored into a product of subdeterminants of the same type, for which all the corresponding m_i are at most $n-1$. The assertion of the theorem then follows from the preceding remarks. We consider the following cases:

Case 1. $m_i = n$; then $\tau_{l(i)} \leqslant \xi_i \leqslant \tau_{l(i)+1}$. Suppose first that $\xi_i = \tau_{l(i)}$. In this case it is not hard to see (with obvious notation) that

$$D = \begin{pmatrix} 0, ..., n-1, \xi_1, ..., \xi_{i-1} \\ \tau_1, ..., \tau_{l(i)} \end{pmatrix} \cdot \begin{pmatrix} 0, ..., n-1, \xi_{i+1}, ..., \xi_r \\ \tau_{l(i)+1}, ..., \tau_N \end{pmatrix}. \tag{3.7}$$

For clarity we make the following remarks. First, if $\xi_i = \tau_{l(i)}$, then the matrix whose determinant is D has block structure, provided the entry with $(\tau_{l(i)} - \xi_i)_+^0$ is set to zero (otherwise, $D = 0$). Secondly, the first n rows of the matrix corresponding to the second determinant in the product are brought into the form shown in (3.5) by elementary row operations.

Similarly, if $\tau_{l(i)} < \xi_i \leqslant \tau_{l(i)+1}$, then, by setting the entry with $(\tau_{l(i)+1} - \xi_i)_+^0$ to one, we get (3.7).

Case 2. $m_i = n+1$; then $\xi_i = \tau_{l(i)} = \tau_{l(i-1)+n+1}$. In this case we delete the column with $\tau_{l(i)}$ and set $m_i = n$. Again, the relevant matrix has block structure and, via elementary row operations on the second block, we get

$$D = \begin{pmatrix} 0, ..., n-1, \xi_1, ..., \xi_{i-1} \\ \tau_1, ..., \tau_{l(i)-1} \end{pmatrix} \cdot \begin{pmatrix} 0, ..., n-1, \xi_{i+1}, ..., \xi_r \\ \tau_{l(i)+1}, ..., \tau_N \end{pmatrix}.$$

The factoring process described in Cases 1 and 2 may now be applied to each of the subdeterminants, as necessary, and the proof is completed as outlined above. ∎

The results of this section reveal something about the zeros of nonnegative perfect splines. The following corollary is a consequence of these results and (2.12).

(3.8) Corollary. *Let $P_0 \in \mathscr{P}(\tau_1, ..., \tau_N)$ be nonnegative in $[a, b]$ and satisfy* (2.1). *Let P_0 have zeros $a < \eta_1 < \cdots < \eta_s < b$ with even multiplicities μ_i $(i = 1, ..., s)$. Then there are points $a < \xi_1 < \cdots < \xi_r < b$ and even integers $2 \leqslant m_i \leqslant n+1$ such that*

$$\{\eta_1, ..., \eta_s\} \subseteq \{\xi_1, ..., \xi_r\},$$

$\mu_i \leqslant m_j$ if $\eta_i = \xi_j$, and the conclusions of (3.3) *and* (3.6) *hold.*

4. An Application of Semi-Infinite Programming

The problem

$$\text{Maximize} \int_a^b s \text{ subject to } s \in S_P \tag{P}$$

may be viewed as a semi-infinite linear programming problem (i.e., a finite number of variables and an infinite number of linear constraints) of the type treated in [2]. This problem has an associated "dual problem"

$$\text{Minimize } y^*(P_0) \text{ subject to } y^*(s) \geqslant \int_a^b s, \ \forall s \in S_0, \ s \geqslant 0$$

$$\text{and } y^*(P_0) \geqslant y^*(s), \ \forall s \in S_0, \ s \leqslant P_0, \tag{D}$$

where y^* is a linear functional on the space spanned by S_0 and P_0. Clearly, the infimum for (P) is no more than the supremum for (D). If

$$\int_a^b \hat{s} = \hat{y}^*(P_0)$$

for some $\hat{s} \in S_P$ and some $\hat{y}^*$ satisfying the constraints, then $\hat{s}$ and $\hat{y}^*$ are optimal for (P) and (D), respectively, and yield the same optimal values. In this case we have

$$\hat{y}^*(P_0) = \hat{y}^*(\hat{s}) = \int_a^b \hat{s}.$$

Let us briefly consider the case $n=2$. The perfect spline P_0 is either positive in (a, b) or it has a finite number of double zeros that do not coincide with knots. If P_0 vanishes at c and at d, with no zeros in (c, d), then we may consider the analogous problem on the interval $[c, d]$ with P_0 set to zero outside of (c, d). In this way the problem (P) splits into a finite number of subproblems for which $P > 0$ on the open interval. Thus, let us assume, without loss of generality, that P is positive on (a, b). In particular, we have $P > 0$ on $[\tau_1, \tau_N]$ and thus the so-called *Slater condition* is satisfied on this interval [2]. It follows that (P) and (D) are solvable and have the same optimal values. We note that, for $n=2$, the functions M_j are "hat functions"—piecewise linear and nonzero in (τ_j, τ_{j+2}) with the peak at τ_{j+1}—and therefore an element $s = \sum_{j=1}^{N-n} c_j M_j \in S_0$ is nonnegative precisely when all of its coefficients c_j are nonnegative. In [2] it is shown that (D) may be expressed as

$$\text{For } \xi_1, ..., \xi_m \in [\tau_1, \tau_N] \text{ minimize } \sum_{i=1}^{m} d_i P_0(\xi_i)$$

$$\text{subject to } d_i \geqslant 0 \text{ and } \sum_{i=1}^{m} d_i M_j(\xi_i) \geqslant \int_a^b M_j = 1 \ (j = 1, ..., N-n). \tag{D$'$}$$

For every optimal solution $\{(d_i, \xi_i)\}_{i=1}^m$ of (D′) and $\{c_j\}_{j=1}^{N-n}$ of (P) we then have

$$\sum_{i=1}^{m} d_i P_0(\xi_i) = \sum_{j=1}^{N-n} c_j = \int_a^b s_0.$$

with $s_0 = \sum_{j=1}^{N-n} c_j M_j$. Moreover, $\{(d_i, \xi_i)\}$ and $\{c_j\}$ are optimal if and only if

$$d_i > 0 \Rightarrow P_0(\xi_i) = \sum_{j=1}^{N-n} c_j M_j(\xi_i) = s_0(\xi_i), \tag{4.1}$$

and

$$c_j > 0 \Rightarrow \sum_{i=1}^{m} d_i M_j(\xi_i) = 1 = \int_a^b M_j. \tag{4.2}$$

Since (4.1) is satisfied at most once in each interval $[\tau_{2i}, \tau_{2i+1}]$, and not at all in (τ_{2i-1}, τ_{2i}), we have $m \leqslant k := (N-n)/2$. Clearly, $P_0 - s_0$ must have a zero in each interval $[\tau_{2i}, \tau_{2i+1}]$, since otherwise s_0 cannot be optimal, and hence $m = k$. Since M_{2i} vanishes outside of (τ_{2i}, τ_{2i+2}), the condition $\sum_{i=1}^{k} d_i M_j(\xi_i) \geqslant 1$ can only be satisfied if $\xi_i \in (\tau_{2i}, \tau_{2i+1})$ and $d_i > 0$ $(i = 1, ..., k)$. From (4.1) we then have $s_0(\xi_i) = P_0(\xi_i)$ (and thus $s_0'(\xi_i) = P_0'(\xi_i)$) for $i = 1, ..., k$. Moreover, $s_0(\xi_i) = P_0(\xi_i) > 0$ implies that at least one of c_{2i-1} and c_{2i} is positive. If both are positive then (4.2) makes it easy to compute ξ_i from the requirement $M_{2i}(\xi_i) = M_{2i-1}(\xi_i)$ (the sum in (4.2) has only one nonzero term).

If all the c_j are positive then the nonnegativity constraints are inactive and s_0 coincides with the best one-sided approximation to P_0 from below (without the nonnegativity restriction). The condition (4.2) then yields a "Gaussian quadrature formula"

$$\sum_{i=1}^{k} d_i s(\xi_i) = \int_a^b s, \ \forall s \in S_0,$$

as has been thoroughly investigated in [7]. In this "unrestricted" problem at most one of a pair c_{2i-1}, c_{2i} can be nonpositive (under the assumption $P_0 > 0$); if one of these is negative then the coefficient with the same index in the solution to (P) must vanish.

For $n > 2$ the situation is more complicated, in part because a nonnegative element of S_0 may have some negative coefficients, so that a simple description of the set $\{s \in S_0 : s \geqslant 0\}$ is not available.

According to the results in [7], the unrestricted one-sided problem has a solution $\hat{s}$ such that $P_0 - \hat{s}$ has exactly k double zeros, $\xi_1, ..., \xi_k$. The spline $\hat{s}$ is uniquely determined by the conditions $\hat{s}^{(j)}(\xi_i) = P_0^{(j)}(\xi_i)$ $(i = 1, ..., k;\ j = 0, 1)$ and a Gaussian quadrature formula exists for S_0, based on $\xi_1, ..., \xi_k$.

In general, the solution s_0 to (P) will differ from $\hat{s}$. Indeed, if P_0 has any zeros of multiplicity greater than two, then $P_0 - s_0$ must have higher than double order zeros, as well, in contrast to $\hat{s}$. Of course, P_0 may be decomposed as above, but only at zeros of even order n or $n+1$. Even if P_0 is positive in (a, b) we cannot expect s_0 and $\hat{s}$ to coincide.

We now continue our study of problems (P) and (D) for $n > 2$. If P_0 is not positive then the Slater condition may not be employed to guarantee a solution for (D) and the equality of the optimal values. Moreover, in general, P_0 may not be decomposed as was the case for $n = 2$. However, a technique applied in [3] may be used to reduce the problem to one for which Slater's condition holds. This is done as follows.

If n is even and P_0 has zeros of order n (which occur between knots) then we split the problem at these points, as with $n = 2$. If n is odd then we can do the same for zeros of order $n+1$ (which occur at knots). Thus, without loss of generality, P_0 has only zeros of even order at most $n-1$, say at $\eta_1 < \cdots < \eta_s$ with multiplicities μ_i. Set

$$\omega(x) := \prod_{i=1}^{s} (x - \eta_i)^{\mu_i}, \qquad \tilde{P}_0 := P_0/\omega, \qquad \tilde{u} := u/\omega,$$

where $u \in S_P$ (hence $\tilde{u}$ is well-defined). The condition $0 \leqslant u \leqslant P_0$ is equivalent to $0 \leqslant \tilde{u} \leqslant \tilde{P}_0$ and now Taylor's Theorem yields $\tilde{P}_0 > 0$ in (a, b) since we have (locally) factored out the zeros and all the μ_i are even. We may therefore consider the problems

$$\text{Maximize} \int_a^b u = \int_a^b \tilde{u}\omega \text{ subject to } 0 \leqslant \tilde{u} \leqslant \tilde{P}_0, \tag{$\tilde{\mathrm{P}}$}$$

with $\tilde{P}_0 > 0$ in (a, b), and

$$\begin{gathered}\text{Minimize } y^*(\tilde{P}_0) \text{ subject to } y^*(\tilde{u}) \geqslant \int_a^b \tilde{u}\omega,\ \forall \tilde{u} \geqslant 0\\ \text{and } y^*(\tilde{P}_0) \geqslant y^*(\tilde{u}),\ \forall \tilde{u} \leqslant \tilde{P}_0.\end{gathered} \tag{$\tilde{\mathrm{D}}$}$$

Here, ω is treated as a weight function. Now Slater's condition is satisfied and thus $(\tilde{\mathrm{D}})$ is solvable and both $(\tilde{\mathrm{D}})$ and $(\tilde{\mathrm{P}})$ have the same optimal values. Moreover, if $\tilde{y}^*$ and $\tilde{u}$ are optimal, then

$$\tilde{y}^*(\tilde{P}_0) = \tilde{y}^*(\tilde{u}) = \int_a^b \tilde{u}\omega.$$

If we define y^* by $y^*(u) := \tilde{y}^*(\tilde{u})$ for $u = \tilde{u}\omega$ then the maximum of $\int_a^b u$ is achieved for some $u_0 \in S_p$ such that

$$u_0^{(j)}(\eta_i) = 0 \qquad (i = 1, ..., s;\ j = 0, ..., \mu_i - 1) \tag{4.3}$$

and there is a linear functional y^* on the subspace

$$S_{0,\mu} \subseteq S_0$$

of splines satisfying (4.3) with

$$y^*(u) \geq \int_a^b u, \qquad \forall u \geq 0, \qquad \text{and} \qquad y^*(P_0) \geq y^*(u), \qquad \forall u \leq P_0.$$

We thus have the following theorem.

(4.4) Theorem. *$s_0 \in S_P$ is a solution to* (P) *iff there is a linear functional Q on $S_{0,\mu}$ such that, for all $s \in S_0$,*

$$Q(s) \geq \int_a^b s, \qquad \forall s \geq 0,$$

$$Q(P_0) \geq Q(s), \qquad \forall s \leq P_0,$$

and

$$Q(P_0) = Q(s_0) = \int_a^b s_0.$$

Proof. If Q satisfies these conditions then for all $s \in S_p$ we have

$$\int_a^b s_0 = Q(s_0) = Q(P_0) \geq Q(s) \geq \int_a^b s,$$

and thus s_0 is optimal. The converse has been proved above.

5. Monosplines and Quadrature Formulas

In the unrestricted one-sided approximation problem alluded to in the previous section, the best approximation yields a linear functional based on the zeros of the error function, which integrates each element of the spline space exactly. Such a functional is called a *quadrature formula* [7, 10]. The error when applying such formulas to functions in $C^m[a, b]$ is given by an integral formula whose kernel is a *monospline*. In this section we investigate the role played by quadrature formulas and monosplines in the solution of our problem.

(5.1) THEOREM. *Suppose that, for $s_0 \in S_P$, $P_0 - s_0$ has no zeros of order $n+1$. Then s_0 is a solution to* (P) *iff there is a linear functional Q such that, for all $s \in S_0$,*

$$Q(s) = \int_a^b s, \qquad \forall s \in S_0,$$
$$Q(P_0) \geqslant Q(s), \qquad \forall s \in S_P, \tag{5.2}$$

and

$$Q(P_0) = Q(s_0).$$

Proof. If (5.2) holds then for all $s \in S_P$ we have

$$\int_a^b s_0 = Q(s_0) = Q(P_0) \geqslant Q(s) = \int_a^b s,$$

and thus s_0 is optimal.

Conversely, let s_0 be the best approximation to P_0 found in (2.12), and let $\xi_1, ..., \xi_r$ be the zeros of $P_0 - s_0$ with multiplicities m_i. The positivity of the determinant in (3.6) implies that the linear system

$$Q(M_l) := \sum_{i=1}^{r} \sum_{j=1}^{m_i} \lambda_{ij} M_l^{(j-1)}(\xi_i) = \int_a^b M_l, \qquad (l = 1, ..., N) \tag{5.3}$$

has a unique solution, provided that $m_i \leqslant n$ $(i = 1, ..., r)$ and that, if $m_i = n$ and $\xi_i = \tau_l$, then for $j = n-1$ a right derivative is taken for $l = l(i)$ and a left derivative is taken for $l = l(i) + 1$. Thus,

$$Q(s) = \sum_{i=1}^{r} \sum_{j=1}^{m_i} \lambda_{ij} s^{(j-1)}(\xi_i) = \int_a^b s, \qquad \forall s \in S_0.$$

The optimality of s_0 implies that for all $s \in S_0$

$$Q(P_0) = Q(s_0) \geqslant \int_a^b s = Q(s),$$

and the proof is complete. ∎

Remark. If $m_i = n+1$ for some i then Q may be defined for the subspace of S_0 consisting of splines without a knot at the zeros of order $n+1$ of $P_0 - s_0$, to which s_0 belongs.

Let

$$M(x) := \int_a^b \frac{(x-t)_+^{n-1}}{(n-1)!}\,dt - \sum_{i=1}^{r} \sum_{j=1}^{m_i} \lambda_{ij}(-1)^j \frac{(x-\xi_i)_+^{n-j}}{(n-j)!} - \sum_{j=0}^{n-1} a_j x^j,$$

where λ_{ij} are the coefficients in the quadrature formula (5.3) and a_j are as yet undefined. The function M is a *monospline* [7, 6]. It follows from results in [7] that for $f \in C^n[a, b]$ with $f^{(j)}(a) = f^{(j)}(b) = 0$ ($j = 0, ..., n-1$) the following "Peano representation" of the error holds

$$\int_a^b f - Q(f) = (-1)^n \int_a^b f^{(n)} M.$$

(5.4) THEOREM. *For appropriate $a_0, ..., a_{n-1}$, the monospline M defined above satisfies*

$$\|M\|_1 = \min \left\{ \int_a^b (P - s) : s \in S_p \right\}.$$

Proof. Suppose for the moment that $m_i \leqslant n$ ($i = 1, ..., r$). Noting that

$$\sum_{i=1}^{r} \sum_{j=1}^{m_i} \lambda_{ij}(-1)^j \frac{(x-\xi_i)_+^{n-j}}{(n-j)!} = Q\left(\frac{(x-\cdot)_+^{n-1}}{(n-1)!}\right),$$

it follows from the standard definition of B-splines as divided differences that, for $l = 1, ..., N$,

$$[\tau_l, ..., \tau_{l+n}] M = \frac{1}{n!} \int_a^b M_l - \frac{1}{n!} Q(M_l) = 0.$$

We have seen that the λ_{ij} are uniquely determined by

$$Q(M_l) = \int_a^b M_l \qquad (l = 1, ..., N-n), \tag{5.5}$$

thus there are unique $\{\lambda_{ij}\}$ and $\{a_j\}$ for which

$$M(\tau_l) = 0 \qquad (l = 1, ..., N). \tag{5.6}$$

Since $M^{(n)} \equiv 1$ in the intervals (ξ_i, ξ_{i+1}), M is *generalized convex* [12, 5, 7] with respect to $S_{\mathbf{m}}^{n,r}(\xi_1, ..., \xi_r)$ (see [12]), which implies

$$(-1)^{N-l} M \geqslant 0 \qquad \text{in} \quad (\tau_l, \tau_{l+1}) \quad (l = 0, ..., N). \tag{5.7}$$

If $m_i = n+1$ for some i then (5.5) holds for the B-splines based only on those τ_l that do not coincide with zeros of order $n+1$ of $P_0 - s_0$, and thus (5.6) is valid for these points as well. Moreover, M is generalized convex on subintervals of $[a, b]$ with endpoints in $\{a, b\} \cup \{\xi_i : m_i = n+1\}$ and, since $m_i = n+1 \Rightarrow \xi_i = \tau_{l(i)}$, with $l(i)$ even, it follows that M changes sign at these points as well. Thus, (5.7) is valid here, too.

We thus have

$$\int_a^b (-1)^n p^{(n)} M = -\int_a^b |M|.$$

Moreover, from the definition of Q and P we get

$$\int_a^b (-1)^n p^{(n)} M = \int_a^b P - Q(P) = Q(P_0) - \int_a^b P_0,$$

hence (5.2) implies

$$\int_a^b |M| = \int_a^b (P_0 - s_0),$$

proving the theorem. ∎

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